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LETTER TO THE EDITOR

Temperley's triangular lattice compact cluster model: exact solution in terms of the q series

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Abstract. Temperley's model of self-supporting stackings of circles in a triangular lattice array against a line wall is solved exactly in terms of q hypergeometric functions. For N circles, the number of different configurations is described by the large-N asymptotic law $A\lambda^N$, with $A = 0.312\,36\ldots$ and $\lambda = 1.735\,66\ldots$.

Recent interest in compact directed cluster models has been stimulated by the fact that some of them are exactly solvable in full detail including the explicit calculation of the generating functions. The only other class of cluster statistics models which have allowed global (beyond critical-point properties) solutions were various random walk models. In this letter we report a new exact solution for a model of 'self-supporting' stackings of circles at a line wall (Temperley 1952). In his paper, Temperley actually formulated several interesting cluster models of N-site lattice animals (connected clusters) near one or more walls. The first model can be defined as follows: N is partitioned in non-increasing combinations

$$N = n_1 + n_2 + \ldots + n_l \qquad 1 \leq l \leq N \tag{1}$$

with $n_i \leq n_{i-1}$. Each group of n_i sites is positioned at the square lattice points:

$$(x, y) = (i-1, 0), (i-1, 1), \ldots, (i-1, n_i - 1).$$

Let c_N denote the number of different animals of this sort (equal to the number of non-increasing partitions of integers), then the entropic free energy per site:

$$s_N \equiv (kT/N) \ln c_N$$

vanishes $\sim 1/\sqrt{N}$, for large N. Further detailed results on the shape of such clusters are available (Temperley 1952, Derrida and Nadal 1984). Note also that $s_N \sim N^{-1/3}$ in 3D.

Another model introduced by Temperley (1952) is obtained by removing the restriction $n_i \leq n_{i-1}$. Thus n_i can take any values consistent with (1). The solution of this model is straightforward (Forgacs and Privman 1986), specifically $s_N \approx kT \ln 2$ is constant for large N. The generating function

$$G(z) = \sum_{N=1}^{\infty} c_N z^N \tag{2}$$

is given by

$$G(z) = \frac{z}{1-2z}. \tag{3}$$

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Thus, with more configurations available, the animals have extensive total entropy. Recently studied compact lattice animal models (Bhat *et al* 1986, Privman and Forgacs 1987, Forgacs and Privman 1987) allow for even larger freedom of lattice animal configurations, yielding $s_N \approx kT \ln \lambda$, with $\lambda > 2$. Note that, for general lattice animal models,

$$c_N \approx AN^{-\theta}\lambda^N \tag{4}$$

for large N, where A and λ are lattice-dependent constants. However, the *compact* animal models discussed in this paragraph share with the simple random walks the property $\theta \equiv 0$.

Temperley (1952) actually considered the 'one tooth' restriction: the allowed configurations satisfy

$$n_1 \leqslant n_2 \leqslant \ldots \leqslant n_j \geqslant \ldots \geqslant n_{l-1} \geqslant n_l$$

where $1 \leqslant j \leqslant l$ (while $1 \leqslant l \leqslant N$ as in (1)). This restriction again results in $s_N \sim 1/\sqrt{N}$. The third model reported by Temperley (1952) as joint work with Onsager is the solid-on-solid model of interfacial properties in 2D. It is outside the scope of our present work: the active field of solid-on-solid models has been reviewed, for example, by Fisher (1986).

Finally, Temperley (1952) mentioned a triangular array packing problem (see details below), the 'one tooth' version of which was solved by Auluck (1951), following Temperley's suggestion (with, again, $s_N \sim 1/\sqrt{N}$). In this letter we study in detail the full version (no 'one tooth' restriction) of this model. Specifically, we calculate analytically the generating function defil ed as in (2).

The definition of the model is illustrated by the open circles in figure 1. The base row of the cluster is continuous. The higher rows can have gaps. However, the cluster must be 'self-supporting': each $y > 0$ site must have both lower-y neighbours occupied. The neighbour sites are defined according to the connectivity of the triangular lattice with spacing equal to the circle diameter, with the x axis along one of the three principal triangular axes. In order to solve the model by the generating function technique, we extend the allowed configurations to include additional $k-1$ base sites along the lattice direction forming 60° with the negative x axis. The case $k = 4$ is illustrated in figure 1. The $k-1 = 3$ full circles are part of the base. Together with the open circles they can 'support' additional sites (full circles in figure 1).

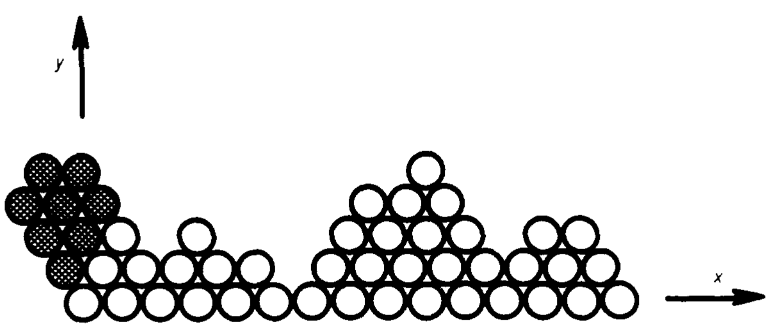

Figure 1. Compact self-supporting packing at the line wall (open circles). Full circles illustrate the three additional 60° base sites (see text) and the sites supported by the extended base.

The number of distinct N-site clusters with $k-1$ 60° root sites will be denoted by $c_{N,k}$. Obviously,

$$c_{N,k}=0 \qquad \text{for} \qquad N<k$$

$$c_{N,N}=1.$$

Let us emphasise that the number of sites forming the x axis part of the base is not fixed. All values consistent with the given k and N are allowed.

We introduce the generating functions

$$F_k(z)=\sum_{N=k}^{\infty} c_{N,k} z^{N-k}.$$

Our objective is to calculate the $k=1$ generating function since

$$G(z)\equiv zF_1(z).$$

However, the extended set $F_k(z)$ is useful because these functions satisfy

$$F_k(z)=1+\sum_{m=1}^{k+1} z^m F_m(z) \qquad k\geq 1. \tag{5}$$

Relations (5) essentially sum up the possible configurations of the 60° row above and to the right of the 60° base row with the origin site included (see figure 1). For example, the 1 in (5) corresponds to no second 60° row (the $N=k$ animal). The zF_1 term corresponds to one site, at $(x,y)=(1,0)$, in the second 60° row, etc. Configurations of the more distant 60° rows are summed up in the appropriate F_m. Except for the $k=1$ 'boundary condition'

$$F_1(z)=1+zF_1(z)+z^2F_2(z) \tag{6}$$

relations (5) can be replaced by their differences:

$$F_{k+1}(z)-F_k(z)=z^{k+2}F_{k+2}(z) \qquad k\geq 1 \tag{7}$$

which form a homogeneous set of equations. Recursions of the general type (7) have been encountered in the theory of q series; see, for example, Adiga *et al* (1985, p 26). (In the present context the role of q is played by the variable z.) However, none of the particular forms there are suitable for our problem (see below). Thus, we devised the q series

$$\phi_k(z)=1+\sum_{n=1}^{\infty}(-1)^n\frac{z^{n(n+k+1)}}{q_n} \qquad q_n\equiv\prod_{j=1}^{n}(1-z^j) \tag{8}$$

which satisfy the recursions

$$\phi_{k+1}(z)-\phi_k(z)=z^{k+2}\phi_{k+2}(z) \tag{9}$$

and are regular at $z=0$ for $k\geq -2$. Note that the general solution of (9) is of the form $\Phi_k(z)=A(z)\phi_k(z)+B(z)\bar{\phi}_k(z)$. However, one can show that the second linearly independent solution, $\bar{\phi}_k(z)$, is singular at $z=0$. Specifically, for any solution with $B(z)\not\equiv 0$ there exists K such that for all $k\geq K$, $\Phi_k(z)/\Phi_K(z)\approx z^{-(k-K)(k+K+1)/2}$ for small z. Let us briefly summarise the results of Adiga *et al* (1985) in the form appropriate for (9). For a positive integer n, we define the solution $\Phi_k^{(n)}(z)$ by $\Phi_{n+1}^{(n)}(z)=1$ and $\Phi_{n+2}^{(n)}(z)=0$. Several $\Phi_k^{(n)}$ with $k<n+1$ and $k>n+2$ can be calculated by using (9). For example, $\Phi_n^{(n)}(z)=1$, $\Phi_{n+3}^{(n)}(z)=-z^{-(n+3)}$, etc. Ramanujan discovered a closed form expression for $\Phi_k^{(n)}$ for $0 \leqslant k \leqslant n$:

$$\Phi_k^{(n)} = \sum_{m=0}^{[(n-k+1)/2]} (-1)^m \frac{z^{m(m+k+1)} q_{n-k-m+1}}{q_m q_{n-k-2m+1}}$$

with $q_0 = 1$. No closed form is available for $k > n+2$. However, already the expression for $\Phi_{n+3}^{(n)}$ indicates that $\Phi_k^{(n)}$ belong to the $B(z) \neq 0$ class of solutions and thus have no direct bearing on the 'physical' problem (5)-(7).

The regular solution $\phi_k(z)$ is sufficient to account for the boundary condition (6). Indeed, since (7) and (9) are homogeneous, we put

$$F_k(z) = A(z)\phi_k(z) \qquad k \geqslant 1$$

and substitute in (6). The result is

$$A(z) = [(1-z)\phi_1(z) - z^2\phi_2(z)]^{-1}.$$

Finally, the generating function for the original problem is represented as

$$G(z) = \frac{z\phi_1(z)}{(1-z)\phi_1(z) - z^2\phi_2(z)}. \tag{10}$$

A simple ratio test for the defining series shows that $\phi_k(z)$ are analytic inside the unit circle in the complex z plane. The form of the denominators, q_n, in (8) suggests that the function $\phi_k(z)$ has a natural boundary at the unit circle. However, the 'physical' singularity of $G(z)$ occurs at $z_c = \lambda^{-1} < 1$ and arises from the first (nearest to the origin) zero of the denominator of (10). It is, therefore, a simple isolated pole as in other solvable compact cluster models with extensive entropy: see (3) and Privman and Forgacs (1987). As already mentioned the simple random walk models have similar singular behaviour.

Thus, the generating functions for cluster statistics models can be simple rational functions or they can involve complicated mathematical objects, like ϕ_k. In fact, the *properties* of such q series have not been investigated in detail in the mathematical literature. Fortunately, in our case the interesting behaviour arises in the region where $\phi_k(z)$ are analytic and in fact easily calculable since the series (8) converge extremely rapidly. Numerical evaluation of (10) confirms the general conclusions. The function

$$\frac{z}{G(z)} = 1 - z - z^2\frac{\phi_2(z)}{\phi_1(z)} \tag{11}$$

is plotted in figure 2. It has a simple zero at

$$z_c = \lambda^{-1} = 0.576\,148\,769\ldots \tag{12}$$

followed by a sequence of poles-zeros (outside the range of figure 2) which seem to accumulate at $z = 1^-$. However, the 'physical' circle of convergence, $|z| < \lambda^{-1}$, is free of singularities (for $G(z)$). The function $G(z)$ can be expanded in powers of z. We generated 60 terms by direct iteration of relations (5):

$$G(z) = z + z^2 + 2z^3 + 3z^4 + 5z^5 + 9z^6 + 15z^7 + 26z^8 + 45z^9 + 78z^{10} + 135z^{11} + \ldots + 126\,487\,349\,805\,945z^{61} + \ldots. \tag{13}$$

The first ten terms shown in (13) were also reproduced by the direct expansion of the exact solution (10). Another consistency check is provided by using all the 60 terms to estimate z_c: consult Privman and Forgacs (1987) for the details of the numerical methods used in such analyses. The value (12) has been confirmed. Note that the effective coordination number' $\lambda = 1.735\,66\ldots$ is less than 2 for this model. The coefficient A in (4) is $A = 0.312\,36\ldots$.

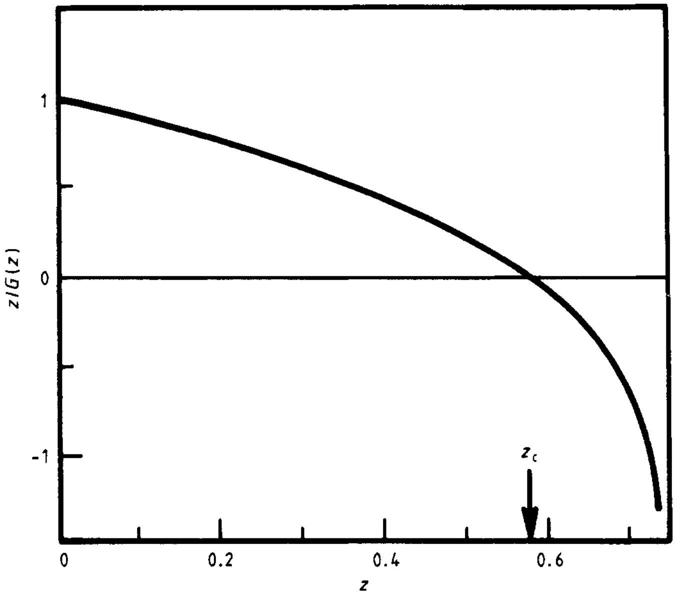

Figure 2. Plot of $z/G(z)$ as a function of z. The physically relevant region is $0 \leq z < z_c$, with positive $G(z)$. At $z_c \simeq 0.576$, the generating function $G(z)$ has a simple pole singularity.

It is interesting to note the relation between $\phi_k(z)$ with the q series occurring in the famous Rogers-Ramanujan identities, which play a role in Baxter's solution of the hard hexagon model (see Baxter 1982, § 14.5). Let

$$H(x, z) = \sum_{m=0}^{\infty} z^{m^2} x^m (q_m)^{-1}.$$

Then $\phi_k(z) = H(-z^{k+1}, z)$. The Rogers-Ramanujan identities are

$$[H(1, z)]^{-1} = \prod_{m=0}^{\infty} (1 - z^{5m+1})(1 - z^{5m+4})$$

$$[H(z, z)]^{-1} = \prod_{m=0}^{\infty} (1 - z^{5m+2})(1 - z^{5m+3}).$$

Possibly, $\phi_k(z)$ possesses such an infinite product representation.

Let us also point out that by the Pincherle theorem (see, e.g., Gautschi 1967), a continued fraction analogue of (11) can be obtained by standard methods. Specifically

$$\frac{\phi_2(z)}{\phi_1(z)} = \cfrac{1}{1 - \cfrac{z^3}{1 - \cfrac{z^4}{1 - \cfrac{z^5}{1 - \dots}}}}.$$

Finally, let us mention the q series:

$$\psi_k(z) = 1 + \sum_{n=1}^{\infty} \frac{z^{n(3n+2k+3)/2}}{q_n^2}$$

satisfying

$$\psi_{k+2}(z)-2\psi_{k+1}(z)+\psi_k(z)=z^{k+3}\psi_{k+3}(z) \tag{14}$$

and regular at $z=0$ for $k\geqslant -3$. Relation (14) arises in the fully directed square lattice compact animal model which has not been solved exactly (Bhat *et al* 1986, Privman and Forgacs 1987). However, the closed form for the second regular (for positive k) linearly independent solution for (14), needed to satisfy the boundary conditions appropriate for the fully directed animal problem, is extremely complicated. This issue will be taken up in a forthcoming publication.

In summary, we have presented a new analytic solution of a compact-animal type model. The distinctive feature of the generating function for this case is the presence of q series which have a rich complex plane structure. However, the 'physical' singularity emerges via a simple mechanism of vanishing denominator in the domain of analyticity of the q series involved. For all the previously solved models with such a 'simple pole' critical behaviour, the generating functions were ratios of polynomials.

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